

MANIFOLDS: HAUSDORFFNESS VERSUS HOMOGENEITY

MATHIEU BAILLIF AND ALEXANDRE GABARD

ABSTRACT. We analyze the relationship between Hausdorffness and homogeneity in the frame of manifolds, not confined to be Hausdorff. We exhibit examples of homogeneous non-Hausdorff manifolds and prove that a Lindelöf homogeneous manifold is Hausdorff.

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1 Introduction

Our purpose here is to analyze the relationship between Hausdorffness and homogeneity in the frame of manifolds. We give the word *manifold* its broadest sense, that is, a topological space locally homeomorphic to the Euclidean space \mathbb{R}^n of a fixed dimension (without assuming the Hausdorff separation axiom).

Recall that a connected Hausdorff manifold M is *homogeneous*, i.e. for each $x, y \in M$, there is a homeomorphism $h : M \rightarrow M$ taking x to y (see [8] or [9], p. 150).

This property is true only under the Hausdorff assumption. Without it, one may well have a non homogeneous manifold, for example the well known *line with two origins*: take two copies of the real line \mathbb{R} and identify all corresponding points of the copies but the origin (Figure 1). This yields a one-dimensional manifold in which the two origins cannot be separated¹. Notice though that a point different from the origins can be separated from any other point, so the manifold is not homogeneous. Another well known example of non-Hausdorff manifold is the *branching line* obtained by identifying the points < 0 in the two copies of \mathbb{R} (Figure 1).

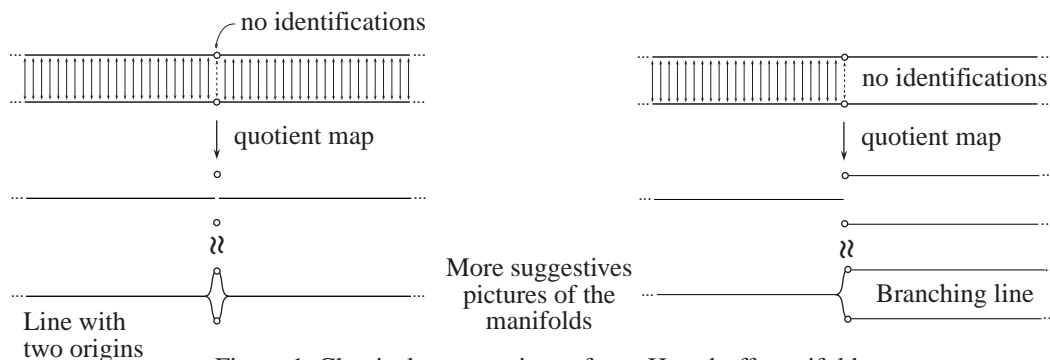


Figure 1. Classical constructions of non-Hausdorff manifolds

One may think that homogeneity is a sufficient condition to characterize, in the realm of manifolds, those which are Hausdorff. We show that this is not the case by exhibiting two examples. The first, called the *complete feather* or *everywhere branching line* F will be discussed in § 2. It was first defined by Haefliger and Reeb in [4], and is constructed by “grafting” lines to all points of a line, and iterating this process indefinitely (see Figure 2).

F is a non-Hausdorff homogeneous 1-manifold but is neither separable² nor Lindelöf³. It furthermore has some interesting contractibility properties: it is contractible but not strongly (i.e.

¹We say that two points of a topological space can be *separated* if there are two disjoint open sets containing one of them each.

²A space is *separable* if it has a countable dense subspace.

³A space is *Lindelöf* if each open cover has a countable subcover.

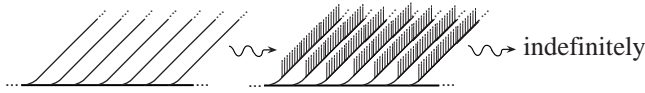


Figure 2. Haeffliger-Reeb's construction of the “plume complète”, (here called complete feather)

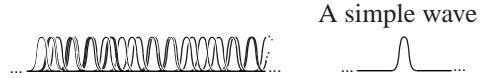


Figure 3. Another example of homogeneous non-Hausdorff manifold

in such a way that the collapsing point stays fixed during the deformation), even though each point has such a strongly contractible neighborhood. This answers a question left open in [3].

Our second example is the *everywhere doubled line* D , a “continuous” version of the line with two origins in which we perform the duplication process at all points (Figure 3 tries to give a representation). D is homogeneous, separable but neither Hausdorff nor Lindelöf. A discussion of this example is the object of § 3. The quest for a Lindelöf example is ruled out by the following:

Theorem 1.1. *A homogeneous Lindelöf manifold is Hausdorff (and therefore metrizable⁴).*

This will be proved in § 4. The proof uses that a manifold is a Baire space⁵ (since a space that is locally Baire⁶ is in fact Baire). Alternatively we can also argue that a (non necessarily Hausdorff) locally compact space is Baire, a fact that is usually proved only for Hausdorff spaces, see for instance [1], but which remains valid in this more general setting (see § 5).

2 The complete feather F

Let us first give a loose description of F . The idea is to start with the usual real line, and to add branches (like in the branching line) at any $x \in \mathbb{R}$. This results in a “hairy line”, with branches at level 1. Then, we continue the process by adding new branches at level 2 to all points in branches in level 1, and so on indefinitely. The resulting space F is a 1-manifold whose homogeneity comes from the fact that we can “flip” a branch at level i with a branch at level $i+1$. It is also contractible.

Now, the formal definition. The underlying set of F is

$$\left\{ s : \begin{array}{l} s = (s_0, \dots, s_n) \text{ for some } n \geq 0, s_i \in \mathbb{R}, \\ \text{and } s_0 < s_1 < \dots < s_{n-1} \leq s_n \end{array} \right\}.$$

Notice that the last inequality is *not* strict. One should interpret the sequences of length 1 as the usual real line, those of length ≤ 2 as the hairy line, and so on. We topologize F with the order topology for the following partial order:

$$(s_0, \dots, s_n) < (t_0, \dots, t_m) \text{ iff } n \leq m, \quad s_i = t_i \text{ for } i = 0, \dots, n-1 \text{ and } s_n < t_n.$$

Notice that (s_0, \dots, s_n) and (s_0, \dots, s_n, s_n) are incomparable and have the same predecessors. F is a “tree” in the sense that the predecessors of any point are totally ordered.

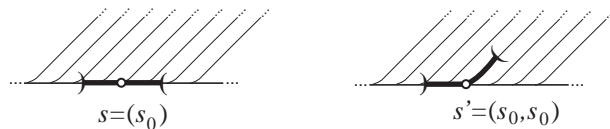


Figure 4. Typical neighborhoods of points according as the point is down or up.

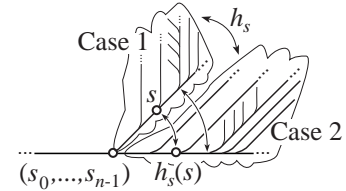


Figure 5. Proving homogeneity.

⁴Recall that a Lindelöf Hausdorff manifold is metrizable; this follows from Urysohn's metrization theorem, since Lindelöf and locally second countable imply second countable.

⁵Every countable intersection of dense open sets is dense.

⁶For any property \mathcal{P} attributable to a space, a space is said to be *locally* \mathcal{P} if each of its points has a neighborhood with the property \mathcal{P} .

Proposition 2.1. *F is a connected homogeneous non-Hausdorff 1-manifold.*

Notice that F is non separable, in fact there is even an uncountable family of pairwise disjoint open sets in F .

Proof. One sees immediately that F is a non-Hausdorff manifold, since the intervals are homeomorphic to \mathbb{R} and the points of the form (s_0, \dots, s_n) and (s_0, \dots, s_n, s_n) cannot be separated. Connectedness is also easy. To see that F is homogeneous, we first show that given any point $s = (s_0, \dots, s_n) \in F$, there is an homeomorphism of F sending s to $(s_n) \in F$. We first consider the map $h_s : F \rightarrow F$ that flips the two “branches” emanating from (s_0, \dots, s_{n-1}) (see Figure 5). It is given by the formula

$$h_s(r) = \begin{cases} (s_0, \dots, s_{n-2}, r_n, r_{n+1}, \dots, r_m) & \text{if } r = (s_0, \dots, s_{n-2}, s_{n-1}, r_n, r_{n+1}, \dots, r_m), \\ (s_0, \dots, s_{n-2}, s_{n-1}, r_{n-1}, r_n, \dots, r_m) & \text{if } r = (s_0, \dots, s_{n-2}, r_{n-1}, r_n, \dots, r_m) \text{ with } r_{n-1} \geq s_{n-1}, \\ r & \text{otherwise.} \end{cases}$$

One sees easily that h_s is an homeomorphism (actually it is an involution). For $s = (s_0, \dots, s_n) \in F$ and $k \leq n$, let $s^{(k)} = (s_0, \dots, s_{n-k})$. Then, $h_{s^{(n-1)}} \circ \dots \circ h_{s^{(1)}} \circ h_s(s) = (s_n)$. To finish the proof it suffices to remark that for $t \in \mathbb{R}$ the map $(s_0, \dots, s_n) \mapsto (s_0 + t, \dots, s_n + t)$ is a homeomorphism. \square

Lemma 2.2. *F is contractible.*

Proof. The idea is to contract all points of the form $(s_0, \dots, s_{n-1}, s_n)$ on the point $(s_0, \dots, s_{n-1}, s_{n-1})$ between time $\frac{1}{n+1}$ and $\frac{1}{n}$. For $x \in \mathbb{R}$ and $t \in [0, 1]$, let $\varphi_t^x : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$\varphi_t^x(y) = \begin{cases} y & \text{if } y < x \\ (1-t)y + tx & \text{if } y \geq x. \end{cases}$$

Then, $\varphi_0^x = \text{id}$, and $\varphi_1^x(y) = x$ for all $y \geq x$. If $s = (s_0, \dots, s_n) \in F$, we let $\phi_t(s) = (s_0, \dots, s_{n-1}, \varphi_t^{s_{n-1}}(s_n))$, so $\phi_1(s) = (s_0, \dots, s_{n-1}, s_{n-1})$. If $s = (s_0, \dots, s_n)$, we write $t' = n(n+1)t - n$ and define

$$h_t(s) = \begin{cases} s & \text{if } n = 0 \text{ or } t \leq \frac{1}{n+1}, \\ \phi_{t'}(s) & \text{if } n \neq 0 \text{ and } t \in [\frac{1}{n+1}, \frac{1}{n}], \\ h_t((s_0, \dots, s_{n-1})) & \text{if } n \neq 0 \text{ and } t > \frac{1}{n}. \end{cases}$$

The definition is implicit, but this causes no problem: we proceed by induction on n . Thus, $h_0 = \text{id}$ and $h_1(s) = (s_0, s_0)$ if $s = (s_0, \dots, s_n)$ with $n > 0$, $h_1((s_0)) = (s_0)$. We then define $h_t((s_0)) = h_t((s_0, s_0)) = (s_0 - t + 1)$ for $t \in [1, 2]$. It is not difficult to see that h_t is continuous and that $h_2(F)$ is included in the sequences of length 1 which are homeomorphic to \mathbb{R} and thus contractible. \square

A space X is *strongly contractible to the point p* if there exists an homotopy $h_t : X \rightarrow X$ such that $h_0 = \text{id}$, $h_1 \equiv p$ and $h_t(p) = p$ for all t . D. Gauld [3] showed that if X is contractible, locally strongly contractible to p and completely regular at p , then X is strongly contractible to p . Further he asked whether “completely regular” could be dropped; the complete feather F gives a counterexample since:

Lemma 2.3. *F is not strongly contractible to any of its points.*

Proof. Call *twins* the pairs of points of F of the form $\{(s_0, \dots, s_n), (s_0, \dots, s_n, s_n)\}$. Any sequence $(s_0, \dots, s_{n-1}, s_n^m)$ ($m \in \mathbb{N}$) with $s_n^m \nearrow s_n$ converges to both twins. Thus, if one of the twins moves, the other must also move. Since any point of F as a twin the result follows. \square

3 The everywhere doubled line D

We can build D either as an inductive limit or with two copies of the line with an exotic topology. We give the latter construction. The underlying set of D is $\mathbb{R} \times \{0, 1\}$. Points of D with zero second coordinate are said “down”, the others “up”. A base for the topology is given by usual open sets downstairs with a finite (eventually zero) number of points removed and lifted upstairs; that is, subsets of the form

$$\mathcal{U}_{O,F} = (O \setminus F) \times \{0\} \cup F \times \{1\},$$

where $O \subset \mathbb{R}$ is open and $F \subset O$ finite. Such subsets will be called *waves*. It is immediate that the waves are closed under finite intersections, we topologize D with the topology given by this base.

Proposition 3.1. *D is a connected non-Hausdorff homogeneous separable 1-manifold.*

Proof. First, it is clear that D is non-Hausdorff, since two points x, y having the same first coordinate cannot be separated. It is also immediate that $\mathbb{Q} \times \{0\}$ is dense in D , and that any wave $\mathcal{U}_{O,F}$ where $O =]a, b[$ is homeomorphic to \mathbb{R} . This proves that X is a separable 1-manifold. Connectedness is also easy.

To check homogeneity, we begin by observing that the translations $t_s : (t, i) \mapsto (t + s, i)$ with $s \in \mathbb{R}$ are clearly homeomorphisms. This settles the case where the two given points have the same second coordinates. If not, use the map exchanging up and down at one value of the abscissas

$$e_s : (t, i) \mapsto \begin{cases} (t, 1 - i) & \text{if } t = s \\ (t, i) & \text{if } t \neq s, \end{cases}$$

which is a homeomorphism, since it acts simply by adding or removing an oscillation to a given wave (or eventually do nothing at all if s is outside from wave’s range). \square

Note. Working with reflections and exchange maps, we even see that X is *involutionally homogeneous* (i.e. the homeomorphism taking x to y can always be chosen to be an involution).

Remarks. As a variant of this construction we can also triple each points of the line. This gives a counterexample to an erroneous claim made by Fuks-Rokhlin who asserted, that any one-dimensional manifold becomes disconnected after ones removes two suitably chosen points (see [2], p. 135).

It is a theorem of Nyikos [5] that a Hausdorff connected manifold of dimension ≥ 1 has cardinality the continuum, the same construction starting with $\mathbb{R} \times \kappa$ for κ any cardinal, shows that (homogeneous) connected manifolds can have arbitrarily large cardinality.

4 Proof of Theorem 1.1

We will prove the following:

Theorem 4.1. *Let X be a topological space which is homogeneous, Lindelöf, locally Hausdorff and Baire. Then X is Hausdorff.*

Theorem 1.1 is then immediate. We will need the following application of Zorn’s Lemma:

Lemma 4.2. *Let X be a locally Hausdorff space. Then for each point $x \in X$, there exists a Hausdorff dense open set U_x containing x .*

Proof of 4.2. Let $x \in X$. Consider \mathfrak{D}_x the set of all Hausdorff open sets containing x , ordered by inclusion. Since X is locally Hausdorff, \mathfrak{D}_x is non-empty. We check that \mathfrak{D}_x is inductive. Let \mathfrak{C} be a totally ordered subset of \mathfrak{D}_x . As usual let $V = \cup_{U \in \mathfrak{C}} U$ be the natural upper bound of \mathfrak{C} . Then V is open, and Hausdorff: given two points in V (say y, z), each of them belongs to some

$V_y, V_z \in \mathfrak{C}$. But since \mathfrak{C} is totally ordered both points belong to one of them (say V_y), and can thus be separated by open sets of V_y , which are also open in X .

By Zorn's lemma, there is U_x maximal in \mathfrak{D}_x . We check its density. So, let Ω be a non-empty open set of X . We can assume that Ω is Hausdorff. If $\Omega \cap U_x = \emptyset$, since Ω and U_x are Hausdorff, so is their union, and thus $U_x \cup \Omega \in \mathfrak{D}_x$, which contradicts the maximality of U_x . \square

Proof of 4.1. By homogeneity, it is enough to prove the existence of a point $x_0 \in X$ which can be separated from each other point $y \in X$.

For all $x \in X$ let $U_x \ni x$ be given by Lemma 4.2. The collection $(U_x)_{x \in X}$ is an open cover of X from which we extract a countable subcover $(U_{x_i})_{i \in \mathbb{N}}$ (by Lindelöfness). Since the U_{x_i} are dense open sets and X is Baire, their intersection $\bigcap_{i \in \mathbb{N}} U_{x_i}$ is dense, and so in particular non-empty. Any point x_0 in this intersection is separable from any other $y \in X$: since the $(U_{x_i})_{i \in \mathbb{N}}$ cover X , y is in U_{x_i} for some $i \in \mathbb{N}$; but so does x_0 , and since U_{x_i} is Hausdorff, x_0 and y can be separated. \square

Remarks. The preceding results raise some problems we found worth mentioning here. Firstly, the Lindelöf condition in Theorem 1.1 is in a sense too strong, since it implies metrizability, and there are non-metrizable Hausdorff manifolds.

Problem 4.3. *In Theorem 1.1, can Lindelöf be replaced by a weaker condition in order to ensure the Hausdorffness but not necessarily the metrizability of the manifold?*

Secondly, the homogeneous non-Hausdorff manifolds F and D both contain an uncountable (closed) discrete subset: Take one point in each branch at level one in F , and all the “up” points in D . So, another problem is:

Problem 4.4. *Is there a homogeneous non-Hausdorff manifold that contains no uncountable (closed) discrete subset, or even stronger that is hereditarily separable⁷?*

(Whether there are non-metrizable *HS* Hausdorff manifolds or not is known to be independent of ZFC. Under *CH*, Rudin-Zenor [6] constructed a non-metrizable *HS* manifold. On the other hand Szentmiklóssy [7] showed that under $MA + \neg CH$, every locally compact *HS* Hausdorff space is *HL*(=hereditarily Lindelöf), and so metrizable if a manifold. The result in [7] is actually stated with compact instead of locally compact, but the above follows by taking the one-point compactification.) A negative answer to the first part of Problem 4.4 would yield that ω_1 -compact⁸ is an answer to Problem 4.3.

5 About the Baire property

We call a space *quasi-compact* if from any open cover one may extract a finite subcover, and *compact* if moreover it is Hausdorff.

Theorem 5.1 (Baire slightly extended). *Let X be a locally compact (not necessarily Hausdorff) space. Then X is a Baire space.*

The following lemma shows that the classical nesting argument used in the proof of Baire's theorem can be applied to X .

Lemma 5.2. *Let X be a locally compact space. Then for each $x \in X$ and each neighborhood $V \ni x$ there is a compact neighborhood $U \subset V$ of x (X is then said to be microcompact).*

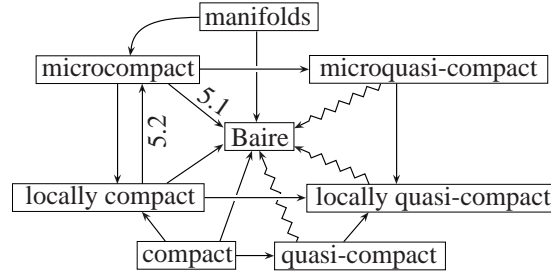
Proof. Let us denote by \mathfrak{V}_x the set of all neighborhoods of x . Recall that a compact space is *regular*, i.e. for each point $x \in X$ and each $V \in \mathfrak{V}_x$, there is a closed set $F \in \mathfrak{V}_x$ with $F \subset V$. Let $V \in \mathfrak{V}_x$, and K be a compact in \mathfrak{V}_x . Then clearly $V \cap K \in \mathfrak{V}_x^K$, i.e. is a neighborhood of x in K . So, there is $F \in \mathfrak{V}_x^K$ a closed set of K with $F \subset V \cap K$. So F is compact, contained in V and it is easy to check that $F \in \mathfrak{V}_x$. \square

⁷In short *HS*, and means that every subspace is separable.

⁸A discrete closed set is at most countable.

Using the classical nesting argument, this implies 5.1.

Remarks. Locally compact cannot be weakened to locally quasi-compact, since a countably infinite set with the finite complement topology is a quasi-compact space which is not Baire. This space is in fact microquasi-compact, that is, has the property given in Lemma 5.2 with ‘compact’ replaced by ‘quasi-compact’. The following chart summarizes the relations between the local compactness properties, and how they stand with respect to Baire (broken arrows mean “does not imply”).



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UNIVERSITÉ DE GENÈVE, SECTION DE MATHÉMATIQUES, 2-4, RUE DU LIÈVRE, CP 240, 1211 GENÈVE 24, SUISSE
E-mail address: baillif@math.unige.ch

UNIVERSITÉ DE GENÈVE, SECTION DE MATHÉMATIQUES, 2-4, RUE DU LIÈVRE, CP 240, 1211 GENÈVE 24, SUISSE
E-mail address: alexandregabard@hotmail.com